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MODELING AND ANALYSIS OF MEMBRANE VIBRATIONS SUBJECTED TO EXTERNAL FORCES

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Abstract. *Vibrational processes in thin-walled structural components play a crucial role in modern mechanical engineering and machine science. Membrane- and plate-like elements are widely used in engineering applications such as acoustic diaphragms, thin mechanical sensors and actuators, lightweight aerospace skins, and elements of machine constructions exposed to dynamic loading. Accurate mathematical models of their oscillatory behavior are essential for predicting dynamic responses, minimizing undesirable vibrations, and ensuring structural reliability.*

This paper investigates the vibrations of membranes subjected to external dynamic forces using the framework of the classical wave equation. The problem is formulated as a second-order partial differential equation with two-point conditions in time. To analyze the dynamic behavior, the differential-symbol method is applied to construct analytical and numerical solutions. The external excitation is introduced as a nonhomogeneous term in the governing equations. The study establishes a class of functions where a unique solution exists and provides explicit formulas for solution construction. An illustrative example is presented for oscillatory processes of an infinite membrane.

The proposed approach contributes to machine science by providing an effective mathematical tool for analyzing vibration modes and dynamic responses of thin-walled components. The results may be applied to improve the design of acoustic devices, structural sensors, and engineering elements subjected to dynamic excitation in mechanical and aerospace systems.

Key words: *membrane oscillation model, wave equation, two-point in time problem, differential-symbol method, wave process, mathematical modelling.*

Introduction.

Vibrational phenomena in structural components represent one of the fundamental challenges in contemporary mechanical engineering and machine science. Thin-walled elements such as membranes and plates are widely used in engineering structures, where their oscillatory behavior significantly affects the efficiency, durability, and reliability of machines. Understanding the dynamics of such components is particularly important in the design of mechanical systems that are exposed to dynamic loads, acoustic excitations, and varying environmental conditions.

Membrane-like structures appear across a wide spectrum of applications: acoustic diaphragms in loudspeakers and measuring devices, lightweight aerospace panels and skins, biomedical and microfluidic membranes, as well as mechanical sensors and actuators. In each of these cases, unwanted or uncontrolled vibrations can lead to reduced performance, increased noise levels, accelerated material fatigue, or even structural failure. Therefore, accurate mathematical models capable of predicting the vibrational response of membranes under external dynamic forcing are of both theoretical and practical interest to mechanical engineering.

The wave equation provides a classical mathematical framework for describing the transverse oscillations of membranes and thin plates. In machine science, this equation is used to analyze stress distributions, displacement fields, vibration modes, and resonance phenomena in mechanical components. However, when external forces act on the system, the governing equations become nonhomogeneous, and their solution requires advanced analytical and numerical techniques. The proper resolution of such problems enables engineers to predict not only the dynamic response of membranes but also to perform diagnostic or inverse analyses, such as identifying material parameters or external loads from observed oscillations.

This paper focuses on the mathematical modeling of membrane vibrations under external dynamic loading. The problem is formulated as a second-order partial differential equation with two-point conditions in time, and the solution is obtained by applying the differential-symbol method. This approach provides a constructive framework for determining unique solutions within a specific class of functions and offers formulas for explicit solution construction. Analytical results are complemented by numerical simulations, illustrating the method's applicability to engineering problems.

The results of this research contribute to the broader field of machine science by providing tools for the analysis of vibrational processes in membrane-like structures. The proposed methodology may serve as a basis for improving the design of acoustic devices, thin-walled mechanical sensors, and structural components in aerospace and automotive engineering. Furthermore, the framework can be extended to investigate more complex oscillatory systems, thereby strengthening the theoretical foundations of mechanical engineering and expanding its application potential.

Every physical or other natural process is described by many parameters. To construct a mathematical model of a process, the most essential of these are selected. Among the first such models, which are reduced to partial differential equations, were the equations of string and membrane vibrations. Modeling processes accompanied by the propagation of vibrations in an elastic medium is based on the application of wave equations. Such equations are used to describe dynamic processes in continuum mechanics, acoustics, hydrodynamics, electrodynamics, and other fields of physics and engineering [1–3].

Classical one-dimensional wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$$

is a mathematical model of longitudinal or transverse vibrations in an elastic medium. Its solution, taking into account the initial and boundary conditions, makes it possible to describe the patterns of disturbance propagation, reflection, and wave interference [4].

Many works [5, 6] consider the extension of the classical model to the case of inhomogeneous media, the presence of dissipation, and external force perturbations. In particular, works [7] investigate the influence of an external source on wave processes modeled using a heterogeneous wave equation with a right-hand side describing the action of the disturbance.

Particular attention should be paid to the approach that consists in reducing the model to a two-point boundary value problem, which allows investigating the spatial structure of wave processes at fixed time parameters [8]. This opens up opportunities for the application of classical analytical methods (e.g., Fourier method, variational approach, functional-analytical methods) to construct exact or approximate solutions.

Problem statement

The propagation of waves in a homogeneous medium under the influence of an external force is generally described by a partial differential equation of the form

$$\frac{\partial^2 U}{\partial t^2} - \gamma^2 \Delta U = f\left(t, x, y\right), \tag{1}$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is Laplace operator. The unknown function $U = U(t, x, y)$ describes a wave process, t is time, γ is phase velocity of wave propagation, $f(t, x, y)$ is an external force acting on an oscillatory system. If the wave equation describes the behavior of sound in a homogeneous medium, then the function $U = U(t, x, y)$ specifies, in particular, the acoustic pressure.

This study examines the set of solutions to the problem associated with equation (1). It is worth noting that equation (1) models a wide range of physical phenomena, including the propagation of forced oscillations in strings and membranes, electromagnetic wave transmission, ocean wave dynamics and seismic wave propagation.

Gravitational waves, light waves, sound waves, as well as oscillations of strings, membranes, and other media, can all be effectively described using the wave equation. Typical examples include the equation of motion for a stretched string:

$$\frac{\partial^2 y(t,x)}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y(t,x)}{\partial t^2}.$$

The propagation speed in such models varies depending on the medium and the nature of the wave. It may represent the speed of light, the speed of sound, or the velocity at which a mechanical displacement (e.g., in a string) travels. Specifically, the speed of mechanical wave propagation in an elastic medium depends on the medium's elastic properties and density.

When modeling processes, conditions for the behavior of an unknown function at certain points in time are often specified. In particular, in the Cauchy problem, the profile of an infinite string and the rate of change of the profile are specified at a single point, i.e., equation (1) is considered with the initial conditions

$$U(0,x) = \varphi(x), \quad \frac{\partial U}{\partial t}(0,x) = \psi(x). \quad (2)$$

The problem (1), (2) is correct and the corresponding homogeneous problem has only a trivial solution. There are mathematical models in which the conditions are set at different points in time. Such conditions are called two-point or multi-point conditions. An example of such conditions is the following

$$U(0,x)=\varphi(x),\ \ U(h,x)=\psi(x). \tag{3}$$

Unlike the Cauchy problem (1), (2), problem (1), (3) is an incorrect two-point problem. In particular, the solutions of equation (1) at that satisfy the conditions

$$U(0,x)=0,\ \ U(h,x)=0, \tag{4}$$

are the functions of the form

$$U_k(t,x)=\sin\frac{\pi kt}{h}\sin\frac{\pi kx}{\gamma h},\ k\in\mathbb{N}.$$

In this paper, we use the differential-symbol method to investigate the process of wave propagation under the action of a force in unbounded domains, described by an unknown function $U(t,x,y)$, which is the solution of equation (1) and satisfies the following two-point conditions in time:

$$U(0,x,y)=0,\qquad U(h,x,y)=0. \tag{5}$$

The homogeneous problem for a partial differential equation of second order in time and of infinite order in spatial variables, was investigated in [9]. The problem for second order in time partial differential equation with two-point conditions was analyzed in [10, 11]. In works [12, 13], the ideas of the differential-symbol method were implemented for solving problems with different conditions for partial differential equations. The behavior of an oscillatory process described by a homogeneous wave equation with homogeneous conditions is studied in [14].

Let's consider the set M is the set of function zeroes

$$\Delta(\nu)=\frac{\sinh\left[\gamma\|\nu\|h\right]}{\gamma\|\nu\|}, \tag{6}$$

where $\|\nu\|=\sqrt{\nu_1^2+\nu_2^2}$,

Let the function $f(t,x,y)$ has the following form

$$f(t,x,y)=\sum_{i=1}^{m}\sum_{j=1}^{n}f_{ij}(t,x,y)e^{\alpha_1x+\alpha_2y+\beta_it},$$

where $\alpha_1,\alpha_2\in C^2\setminus M$, $\beta_1...,\beta_m\in C^2\setminus M$, $f_{11}(t,x,y),...,f_{nm}(t,x,y)$ are arbitrary nonzero polynomials with complex coefficients of variables t and x.

The problem (1), (5) has an unique solution in the class of quasi-polynomials, and this solution can be found by the formula [15]

$$U(t,x,y)=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left\{F(t,\lambda,\nu)e^{\nu_1x+\nu_2y}\right\}\Big|_{\lambda=0,\nu=O},\quad(7)$$

where $F(t,\lambda,\nu)=\dfrac{e^{\lambda t}-T_1(t,\nu_1,\nu_2)-e^{\lambda h}T_2(t,\nu_1,\nu_2)}{L(\lambda,\nu_1,\nu_2)}$,

$$T_1(t,\nu_1,\nu_2)=\frac{1}{\Delta(\nu)}\frac{\sinh\left[\gamma\|\nu\|(h-t)\right]}{\gamma\|\nu\|},\quad T_2(t,\nu_1,\nu_2)=\frac{1}{\Delta(\nu)}\frac{\sinh\left[\gamma\|\nu\|t\right]}{\gamma\|\nu\|}.$$

Let's prove that the function of the form (7) is the solution of the equation (1).

$$\left(\frac{\partial^2}{\partial t^2}-\gamma^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)U(t,x,y)=$$

$$=\left(\frac{\partial^2}{\partial t^2}-\gamma^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left\{F(t,\lambda,\nu)e^{\nu_1x+\nu_2y}\right\}\Big|_{\lambda=0,\nu=O}=$$

$$=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left(\frac{\partial^2}{\partial t^2}-\gamma^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)\left\{F(t,\lambda,\nu)e^{\nu_1x+\nu_2y}\right\}\Big|_{\lambda=0,\nu=O}=$$

$$=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\frac{\partial^2}{\partial t^2}\left\{F(t,\lambda,\nu)e^{\nu_1x+\nu_2y}\right\}\Big|_{\lambda=0,\nu=O}-$$

$$-\gamma^2\left(f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)\left\{F(t,\lambda,\nu)e^{\nu_1x+\nu_2y}\right\}\Big|_{\lambda=0,\nu=O}$$

Taking into account following equality

$$\frac{\partial^2T_1(t,\nu_1,\nu_2)}{\partial t^2}=\gamma^2T_1(t,\nu_1,\nu_2),\quad\frac{\partial^2T_2(t,\nu_1,\nu_2)}{\partial t^2}=\gamma^2T_2(t,\nu_1,\nu_2),$$

we get

$$\left(\frac{\partial^2}{\partial t^2}-\gamma^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)U(t,x,y)=f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu_1},\frac{\partial}{\partial\nu_2}\right)\left\{e^{\lambda t+\nu_1x+\nu_2y}\right\}\Big|_{\lambda=0,\nu=O}$$

$$=f(t,x,y).$$

It is easy to verify that function (7) satisfies conditions (5). The uniqueness of the solution to problems (1), (5) can be proved by contradiction.

Let's consider two-point problem for following condition

$$(0,x,y)=0, \quad U(1,x,y)=0. \tag{8}$$

We write the functions $\Delta(\nu_1,\nu_2)$, $T_1(t,\nu_1,\nu_2)$, $T_2(t,\nu_1,\nu_2)$ and the set M for the problem (1), (8): $\Delta(\nu)=\dfrac{\sinh\left[\gamma\|\nu\|\right]}{\gamma\|\nu\|}$, $M=\left\{\nu\in C^2:\|\nu\|^2=-\left(\dfrac{\pi k}{\gamma}\right)^2, k\in N\right\}$,

$$T_1(t,\nu)=\frac{\sinh\left[\gamma\|\nu\|(1-t)\right]}{\gamma\|\nu\|}, \quad T_2(t,\nu)=\frac{\sinh\left[\gamma\|\nu\|t\right]}{\gamma\|\nu\|}.$$

The function $F(t,\lambda,\nu_1,\nu_2)$ for problem (1), (8) has the form:

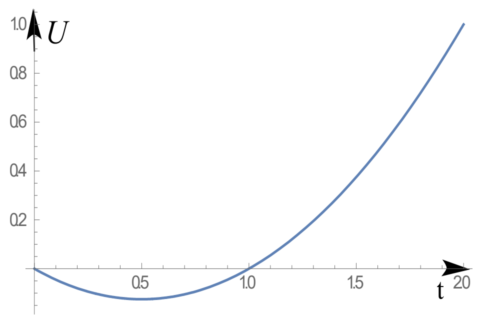
$$F(t,\lambda,\nu_1,\nu_2)=\frac{e^{\lambda t}-T_1(t,\nu_1,\nu_2)-e^{\lambda h}T_2(t,\nu_1,\nu_2)}{L(\lambda,\nu_1,\nu_2)}.$$

Let's find unknown function $U(t,x,y)$, if a constant external force $f(t,x,y)=1$ acts on the oscillatory system. Since $\Delta(0,0)=1\neq 0$, then the solution of problem (1), (8) can be found by formula (7):

$$U(t,x,y)=\left\{F(t,\lambda,\nu_1,\nu_2)e^{\nu_1 x+\nu_2 y}\right\}\Big|_{\lambda=0,\nu=O}=$$

$$=\left.\frac{e^{\lambda t}-\dfrac{1}{\sinh\left[\gamma\|\nu\|\right]}\left(\sinh\left[\gamma\|\nu\|(1-t)\right]+e^{\lambda}\sinh\left[\gamma\|\nu\|t\right]\right)}{\lambda^2-\gamma^2\|\nu\|^2}e^{\nu_1 x+\nu_2 y}\right|_{\lambda=0,\nu=O}=$$

$$=-\left.\frac{\left(\sinh\left[\gamma\|\nu\|(1-t)\right]+\sinh\left[\gamma\|\nu\|t\right]\right)}{\left(1-\gamma^2\|\nu\|^2\right)\sinh\left[\gamma\|\nu\|\right]}e^{\nu_1 x+\nu_2 y}\right|_{\nu=O}=\frac{1}{2}t(t-1).$$



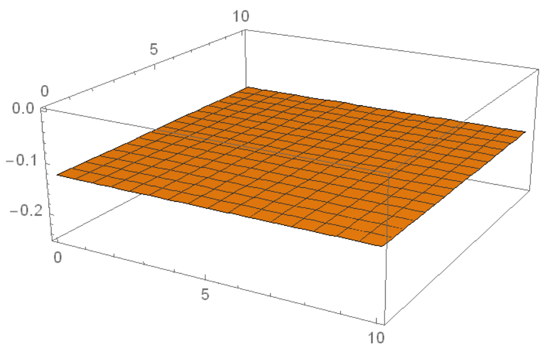


Fig.1a) The graph of the function $U(t)$ **Fig.1б) The function $U(0.5,x,y)$**

Therefore, if a constant external force acts on the oscillatory system described by problem (1), (8), then the unknown function depends only on time (Fig. 1a, b)).

Let's find unknown function $U(t,x,y)$, if an external force $f(t,x,y)=te^{x+y}$ acts on the oscillatory system. Since $\Delta(0,0)=1\neq 0$, then the solution of the problem (1), (8) we find by formula (7):

$$U(t,x,y)=\frac{\partial}{\partial\lambda}\left\{F(t,\lambda,\nu)e^{\nu_1 x+\nu_2 y}\right\}\Big|_{\lambda=0,\nu=O}=$$

$$=\frac{\partial}{\partial\lambda}\left\{\frac{e^{\lambda t}-\frac{1}{\sinh\left[\gamma\|\nu\|\right]}\left(\sinh\left[\gamma\|\nu\|(1-t)\right]+e^{\lambda}\sinh\left[\gamma\|\nu\|t\right]\right)}{\lambda^2-\gamma^2\|\nu\|^2}e^{\nu_1 x+\nu_2 y}\right\}\Bigg|_{\lambda=0,\,\nu_1=1,\nu_2=1}=$$

$$=-\frac{\left(t\sinh\left[\gamma\|\nu\|\right]-\sinh\left[\gamma\|\nu\|t\right]\right)}{\gamma^2\|\nu\|^2\sinh\left[\gamma\|\nu\|\right]}e^{\nu_1 x+\nu_2 y}\Bigg|_{\nu_1=1,\nu_2=1}=-\frac{1}{2\gamma^2}\left(t-\frac{\sinh\left[\sqrt{2}\gamma t\right]}{\sinh\left[\sqrt{2}\gamma\right]}\right)\mathrm{e}^{x+y}.$$

In the case an external force $f(t,x_1,x_2)=te^{x_1+x_2}$ acts on the oscillatory system, the solution for problem (1), (8) describes the oscillatory process at any given moment in time. In particular, at moments $t=0,5$ and $t=2$ we obtain, respectively:

$$U(0,5,\,x_1,x_2)=-\frac{1}{2\gamma^2}\left(\frac{1}{2}-\frac{\sinh\left[\sqrt{2}\gamma/2\right]}{\sinh\left[\sqrt{2}\gamma\right]}\right)\mathrm{e}^{x+y},$$

$$U(2,\,x,y)=-\frac{1}{2\gamma^2}\left(2-\frac{\sinh\left[2\sqrt{2}\gamma\right]}{\sinh\left[\sqrt{2}\gamma\right]}\right)\mathrm{e}^{x+y}.$$

Future research may focus on extending the developed mathematical model by incorporating additional physical factors, such as material inhomogeneity, anisotropy, damping effects, nonlinearity, and more complex forms of external excitation — including impulsive, stochastic, or spatially nonuniform forces.

Special attention may be devoted to the application of the proposed approach to inverse problems, such as identifying unknown external forces or system parameters from known system states at two different time moments. This opens opportunities for use in control, diagnostics, and signal reconstruction in distributed parameter systems.

Conclusions

The process of wave propagation under the action of an external dynamic force in an unbounded spatial domain, where the values of the unknown function are specified at two distinct moments in time, is mathematically modeled by a two-point in time problem for a non-homogeneous wave equation. This type of formulation arises naturally in physical systems where the state of the medium is known at both the initial and final time points, and the goal is to determine the behavior of the system in the intermediate time interval under the influence of external forcing.

To solve this problem, the differential-symbol method is employed. Within this framework, the necessary and sufficient conditions for the existence and uniqueness of the solution are rigorously derived, ensuring the mathematical well-posedness of the problem. Furthermore, an explicit constructive formula for the solution is obtained, which incorporates the effects of the external force and the specified boundary data in time.

This approach not only provides a theoretical foundation for analyzing such wave processes but also opens the way for efficient numerical implementation and deeper insight into the dynamics of forced oscillations in unbounded media. The results are applicable to a wide range of problems in acoustics, continuum mechanics, and other fields involving wave propagation in infinite or semi-infinite domains.

References:

1. Folland, G. (1995). Introduction to Partial Differential Equations, Princeton University Press, Princeton.

2. Wazwaz, A.-M. (2009). Partial Differential Equations and Solitary Waves Theory. Springer.

3. Li, B. (2008). Wave Equations for Regular and Irregular Water Wave Propagation. Journal of Waterway, Port, Coastal, and Ocean Engineering, vol. 134, issue 2, pp.121–142.
DOI: http://dx.doi.org/10.1061/(ASCE)0733-950X(2008)134:2(121)

4. Friedman, A. (1961). The wave equation for differential forms. Pacific Journal

of Mathematics, vol.11, issue 4, pp.1267–1279.
DOI: 10.2140/pjm.1961.11.1267

5. Alkhadhr S., Almekkawy M. (2023). Wave Equation Modeling via Physics-Informed Neural Networks: Models of Soft and Hard Constraints for Initial and Boundary Conditions. Sensors, vol. 23, issue 5, pp. 2792.
DOI: 10.3390/s23052792

6. He X., Wang Q., Zhou Y., Huang J., Huang X. (2024). An Effective Discontinuous Galerkin Method for Solving Acoustic Wave Equations in Heterogeneous Media. Journal of Geophysics and Engineering, issue 1, pp. 105–118.
DOI: 10.1093/jge/gxae119

7. Chorfi S.E., El Guermai G., Maniar L., Zouhair W. (2024). Lipschitz stability for an inverse source problem of the wave equation with kinetic boundary conditions. arXiv:2402.12902.
https://arxiv.org/abs/2402.12902v1

8. Hoonhout, D. & Urzúa-Torres, C. (2025). Towards Stable Second-Kind Boundary Integral Equations for Transient Wave Problems.
DOI:10.48550/arXiv.2502.01374

9. Malanchuk, O., Nytrebych, Z., (2017). Homogeneous two-point problem for PDE of the second order in time variable and infinite order in spatial variables. Open Mathematics, vol.15, issue 1, pp.101–110.
DOI: 10.1515/math-2017-0009

10. Nytrebych, Z. M., Malanchuk, O. M. (2017). The differential-symbol method of solving the problem two-point in time for a nonhomogeneous partial differential equation. Journal of Mathematical Sciences, vol. 227, issue 1, pp. 68–80.
DOI: 10.1007/s10958-017-3574-2

11. Nytrebych, Z. M., Malanchuk, O. M. (2017). The differential-symbol method of solving the two-point problem with respect to time for a partial differential equation. Journal of Mathematical Sciences, vol. 224, issue 4, pp. 541–554.
DOI: 10.1007/s10958-017-3434-0

12. Nitrebich, Z.M. (1996). An operator method of solving the Cauchy problem

for a homogeneous system of partial differential equations. Journal of Mathematical Sciences, vol. 81, issue 6, pp. 3034–3038.
DOI: 10.1007/bf02362589

13. Kalenyuk, P.I., Kohut, I.V., Nytrebych, Z.M. (2012). Problem with integral condition for a partial differential equation of the first order with respect to time. Journal of Mathematical Sciences. vol.181, issue 3, pp. 293–304.
DOI: 10.1007/s10958-012-1081-z

14. Nytrebych, Z.M., Malanchuk, O.M., Il'kiv, V.S., Pukach, P.Ya. (2017). Homogeneous problem with two-point conditions in time for some equations of mathematical physics. Azerb. Journal of Mathematics, vol.7, issue 2, pp. 180–196.

15. Nytrebych, Z., Malanchuk, O., Il'kiv, V., Pukach, P. (2017). On the solvability of two-point in time problem for PDE. Italian Journal of Pure and Applied Mathematics, vol.38, pp.715–726.

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